Published by Institute of Physics Publishing for SISSA

Received: *February 5, 2007*
Revised: *May 29, 2007*
Accepted: *December 18, 2007*
Published: *December 21, 2007*

NS-NS fluxes in Hitchin's generalized geometry

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Abstract: The standard notion of NS-NS 3-form flux is lifted to Hitchin's generalized geometry. This generalized flux is given in terms of an integral of a modified Nijenhuis operator over a generalized 3-cycle. Explicitly evaluating the generalized flux in a number of familiar examples, we show that it can compute three-form flux, geometric flux and non-geometric Q-flux. Finally, a generalized connection that acts on generalized vectors is described and we show how the flux arises from it.

Keywords: String Duality, Flux compactifications.

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1. Introduction

Generalized (complex) geometry, developed by Hitchin [1 – 3] and Gualtieri [4], has emerged as a useful framework for describing new string compactifications. It naturally includes a large class of vacua known as generalized Calabi-Yau manifolds [5 – 11], and also gives a more elegant description of so-called non-geometric spaces or T-folds [12 – 18].

A strength of this formalism is that it is naturally covariant under T-duality provided one dualizes along a U(1)-isometry direction. While the action of T-duality on the sugra fields, given by the Buscher rules, is very complex [19 – 21], the corresponding transformations in generalized geometry are quite simple.

One of the interesting features of generalized geometry is that the metric and B-field are no longer considered the fundamental objects. Rather, it is only the combination of g and B [22],

$$\mathsf{G} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}, \tag{1.1}$$

that enters into the formalism. This unification of B and g is very natural in a T-duality covariant formalism since B and g mix under T-duality.

However, having grouped g and B together into a single generalized object, G, we are left with a puzzle: What is the generalized version of NS-NS 3-form flux, $H = dB$? In particular, we would like to find a generalized analogue of

$$\int_\Sigma H\,, \tag{1.2}$$

where Σ is a 3-cycle and $H = dB$ is the NS-NS 3-form flux. Since generalized geometry is covariant under T-duality, the generalized version of (1.2) should also capture its various T-duals. Under T-duality, 3-form flux is mapped to so-called geometric flux, which is given by the first Chern-class of a circle bundle [23, 24, 14, 25]. Applying T-duality once more, the flux becomes the somewhat obscure non-geometric flux [14, 18, 26 − 30, 15, 17, 11, 31] or Q-flux in the parlance of [9, 10].

The purpose of this paper is to argue that the generalized analogue of H, which we denote H, is a slight modification of the Nijenhuis operator given in [4]. Just as H, being a three-form, can be defined by its action on vectors, $H(V_1, V_2, V_3)$, we define H by its action on *generalized* vectors $\mathsf{V} \in T \oplus T^*$,

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = -\mathsf{Nij}(\widetilde{\mathsf{V}}_1, \widetilde{\mathsf{V}}_2, \widetilde{\mathsf{V}}_3)\,, \tag{1.3}$$

where Nij is the Nijenhuis operator and we define $\widetilde{\mathsf{V}} = \mathsf{GV}$.

Given this definition of H, we next define what it means to integrate it over a 3-cycle Σ. In the case of ordinary three-flux this is a trivial step, since we need only to pull the 3-form H back to the three-cycle and integrate it. In the case of the generalized flux, we will need some extra data on our three-cycle in order to define integration. This extra data will amount to specifying an involutive maximal isotropic subbundle $\Omega \in T \oplus T^*$. Roughly speaking, Ω is needed to define the "frame" in which one is defining the flux. When $\Omega = T^*$, our formulas will reduce to just the ordinary formula for H-flux. When Ω includes vectors as well as forms, we will measure geometric and non-geometric fluxes. We will call the combination of Σ and Ω a *generalized 3-cycle*, $\mathsf{\Sigma}$.

Finally, we will give a prescription for integration of H over our generalized three-cycle $\mathsf{\Sigma}$. As we will see, this is the most subtle part of the story. Because generalized geometry is naturally covariant under T-duality, one ends up needing a prescription for integration over a dual direction. Such a notion of integration can only be defined when the 3-cycle has various isometries and we will need to put certain restrictions on the form of the 3-cycles. In the end, we will only give a partial prescription for this integration, but our definition will be sufficiently general to see that the generalized flux H captures all of the T-duals of H-flux.

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Having defined a generalized notion of H-flux, we next present an additional motivation for the formula (1.3). Recalling that H-flux often arises as the torsion of connection, we construct a *generalized* connection D on $T \oplus T^*$. This connection is not a connection in the standard sense, since it allows one to differentiate along T-dual directions. Constructing what seems to be the natural analogue of the torsion of the generalized connection, we find that it vanishes. However, we show that a certain torsion-like antisymmetric object built from the connection reproduces the generalized flux formula (1.3).

The organization of this paper is as follows: We begin with a review of generalized geometry in section 2. In section 3 we define the generalized flux, which we illustrate in section 4 in several examples. In section 5 we demonstrate a relationship between the generalized flux and the generalized connection. Finally, we conclude in section 6 with a discussion of some open problems and future directions.

2. Review of generalized geometry

One motivation for introducing generalized geometry is that it is a formalism in which T-duality acts in a simple way. While this formalism has been developed recently by Hitchin [1−3] and Gualtieri [4], it has its roots in the older work of Duff [22] and Tseytlin [32]. In this section, we give an introduction to the subject which focuses on its relationship with T-duality of the string worldsheet. The reader familiar with the generalized literature is warned that this discussion is atypical and is not meant to explain the mathematical origins of generalized geometry which are given, for example, in [4].

2.1 T-duality and generalized geometry

We begin with a review of how T-duality acts at the level of the classical string action. Consider a string propagating on a d-dimensional Euclidean manifold M, with metric $g_{\mu\nu}$ and B-field $B_{\mu\nu}$;

$$S = \frac{1}{2}\int g_{\mu\nu} dX^\mu \wedge *dX^\nu + B_{\mu\nu} dX^\mu \wedge dX^\nu \ . \tag{2.1}$$

When $g_{\mu\nu}$ and $B_{\mu\nu}$ do not depend on the coordinates, X^μ, we can rewrite the action as

$$S = \frac{1}{2}\int g_{\mu\nu} V^\mu \wedge *V^\nu + B_{\mu\nu} V^\mu \wedge V^\nu + 2\, d\hat{X}_\mu \wedge V^\mu \ . \tag{2.2}$$

To recover the original action (2.1), one integrates out $\hat{X}$, which imposes $dV = 0$. Since every closed 1-form is locally exact,[1] we may replace $V = dX$, yielding (2.1).

If instead, one integrates out V, one gets a new sigma model in terms of $\hat{X}$ that is the T-dual of the original model with a new $\hat{g}$ and $\hat{B}$ that are related to g and B by the Buscher rules [20, 21]. One also discovers the on-shell relationship between the coordinate X^μ and its T-dual, $\hat{X}_\mu$;

$$d\hat{X}_\mu = g_{\mu\nu} * dX^\nu + B_{\mu\nu} dX^\nu \ . \tag{2.3}$$

[1] Worrying about global issues on the worldsheet reveals the standard exchange of winding and momentum modes and requires that the coordinates X^μ be periodic.

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As noted by Duff [22], if we combine dX and $d\hat{X}$ into a vector,

$$\begin{pmatrix} dX^\mu \\ d\hat{X}_\mu \end{pmatrix}, \tag{2.4}$$

then T-duality acts in a very simple way by exchanging elements of the top with elements of the bottom.

Loosely speaking, in generalized geometry, we can define a *generalized vector* to be an element $\mathsf{V} \in T \oplus T^*$,

$$\mathsf{V} = \begin{pmatrix} V^\mu \\ \omega_\mu \end{pmatrix}, \qquad V \in T, \qquad \omega \in T^*, \tag{2.5}$$

which transforms under T-duality, as well as diffeomorphisms and gauge transformations of B, in the same way as (2.4). Since it will appear often, we define $\mathsf{E} = T \oplus T^*$.[2]

Since V is a direct sum of a vector and a 1-form, we will also write V as a formal sum of a vector and a 1-form,

$$\mathsf{V} = V + \omega, \tag{2.6}$$

as is standard in the generalized literature [1–4].

2.2 Symmetries of $\mathsf{E} = T \oplus T^*$

We now describe explicitly how various gauge transformations act on E. For readers familiar with the generalized literature, we note that we are describing spacetime symmetries and not the symmetries of the frame bundle, which can be arbitrary elements of $O(d,d)$.

We've already seen that, under T-duality, we just exchange forms and vectors. For example, if we take our spacetime to be 2-dimensional, and T-dualize along the 1-direction, we would map

$$\mathsf{V} = \begin{pmatrix} V^1 \\ V^2 \\ \omega_1 \\ \omega_2 \end{pmatrix} \xrightarrow{\mathrm{T}_1} \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ V^2 \\ V^1 \\ \omega_2 \end{pmatrix}. \tag{2.7}$$

It is important to remember that T-duality will only be an allowed transformation when the direction we are T-dualizing along is a $\mathrm{U}(1)$ isometry. As we will see later, this will require that the generalized vectors we are dualizing be independent of the $\mathrm{U}(1)$-isometry direction. Whenever we speak of an object transforming covariantly under T-duality, we will always mean this restricted sense. T-dualities thus form a discreet set of global symmetries.

We also have two kinds of local symmetries, diffeomorphisms and gauge transformations of B. Diffeomorphisms act in the natural way on the vector and form indices. Explicitly, if we transform coordinates from X^μ to $X^{\mu'}$ and define $M^{\mu'}{}_\mu = \partial X^{\mu'}/\partial X^\mu$ then V transforms as

$$\mathsf{V} \to \begin{pmatrix} (M^{-1})^{\mathrm{t}} & 0 \\ 0 & M \end{pmatrix} \mathsf{V}\,. \tag{2.8}$$

[2] In the presence of a non-trivial B-field, one can only split E into $T \oplus T^*$ locally as T^* is twisted by a gerbe [3]. In the case of non-geometric spaces, both T and T^* may be twisted.

Under gauge transformations $B \to B + d\lambda$,[3] we can see from (2.3) that $d\hat{X}_\mu \to d\hat{X}_\mu + (d\lambda)_{\mu\nu} dX^\nu$. Thus, for the generalized vector (2.5), we should shift $\omega_\mu \to \omega_\mu + (d\lambda)_{\mu\nu} V^\nu$. This can be written in matrix form as

$$\mathsf{V} \to \begin{pmatrix} 1 & 0 \\ d\lambda & 1 \end{pmatrix} \mathsf{V} \,. \tag{2.9}$$

Another standard notation for a gauge transformation of B, which uses the notation introduced in (2.6) and is common in the generalized literature, is to write

$$e^{\delta B}(V + \omega) = V + \omega + i_V \delta B \,, \tag{2.10}$$

where $\delta B = d\lambda$, and we consider δB to be acting from the left by contracting indices with vectors. Note that, as is standard, for a form $\rho_{\mu_1 \ldots \mu_n}$, we define $(i_V \rho)_{\mu_2 \ldots \mu_n} = V^{\mu_1} \rho_{\mu_1 \ldots \mu_n}$.

2.3 The canonical inner product and the metric G

The diffeomorphisms, B-transformations and T-duality transformations are all symmetries of the canonical inner product given by

$$\langle \mathsf{V}_1, \mathsf{V}_2 \rangle = \langle V_1 + \omega_1, V_2 + \omega_2 \rangle = \omega_1(V_2) + \omega_2(V_1) \,, \tag{2.11}$$

where $\omega(V) = V^\mu \omega_\mu$. This metric has signature (d, d) and is thus invariant under local rotations in $O(d, d)$. Note that the full local $O(d, d)$ symmetry is only partially generated by (2.7), (2.8) and (2.9). The $\mathsf{U}(1)$-isometry condition on the T-duality transformation, as well as the requirement that the B-transformations are pure-gauge, put various restrictions on the allowed symmetries. These extra conditions are required for the Courant bracket (to be introduced presently) to transform covariantly.

A subbundle $\Omega \in \mathsf{E}$ on which the canonical metric vanishes is said to be isotropic. It is said to be maximally isotropic if its dimension is half that of E. Simple examples of maximally isotropic subbundles are T^* and T.

So far, we have been ignoring the fact that dX and $d\hat{X}$ are not independent fields, and it might seem that we need to impose (2.3) to project out some of the generalized vectors. In fact, however, the condition (2.3) naturally defines two conditions $d\hat{X}_\mu = \pm g_{\mu\nu} dX^\nu + B_{\mu\nu} dX^\nu$ depending on whether we are studying *right-moving* or *left-moving* fields. In terms of a generalized vector V, of the form (2.5), these two conditions $\omega_\mu = \pm g_{\mu\nu} V^\nu + B_{\mu\nu} V^\nu$ restrict V to be in one of two subspaces $\mathsf{C}^\pm \subset \mathsf{E}$;

$$\mathsf{C}^\pm \equiv \operatorname{span} \left\{ \begin{pmatrix} V^\mu \\ \pm g_{\mu\nu} V^\nu + B_{\mu\nu} V^\nu \end{pmatrix} \,\middle|\, V^\mu \in T \right\} . \tag{2.12}$$

Conveniently, these two spaces are orthogonal under the inner product (2.11) and satisfy $\mathsf{C}^+ \oplus \mathsf{C}^- = \mathsf{E}$. They therefore define a *splitting*.

[3] We should also include the large gauge transformations, $B \to B + \delta B$ which are closed, but not exact, and are in integer cohomology.

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This splitting can be encoded by a matrix G which has eigenvalues $+1$ for elements in C^+ and eigenvalues of -1 for elements of C^-. Explicitly,

$$\mathsf{G} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} . \tag{2.13}$$

Note that $\mathsf{G}^2 = 1$, which follows from the fact that its eigenvalues are ± 1. Heuristically, G should be thought of as the analogue of the Hodge star, $*$, on the world sheet. In generalized geometry, we never speak of the metric and B-field separately, it is only the combination (2.13) which enters the story. If Λ is some combination of diffeomorphisms, B-transformations and T-dualities, G transforms as $\mathsf{G} \to \Lambda\mathsf{G}\Lambda^{-1}$.

Using G one can define a positive-definite inner product on E;

$$\mathsf{G}(\mathsf{A},\mathsf{B}) = \langle \mathsf{A}, \mathsf{G}\mathsf{B}\rangle = \langle \mathsf{G}\mathsf{A}, \mathsf{B}\rangle \ . \tag{2.14}$$

This inner product often acts as the generalized version of the metric g.

2.4 The Courant-bracket and the Nijenhuis operator

A basic object in generalized geometry, whose properties are discussed in detail in [4], is the Courant-bracket,

$$[V_1 + \omega_1, V_2 + \omega_2]_C = [V_1, V_2]_L + \mathcal{L}_{V_1}\omega_2 - \mathcal{L}_{V_2}\omega_1 - \frac{1}{2}(d(i_{V_1}\omega_2) - d(i_{V_2}\omega_1)) , \tag{2.15}$$

where $[V_1, V_2]_L$ is the Lie-bracket of two vector fields and $\mathcal{L}_V = i_V d + d i_V$ is the Lie-derivative. As with the other objects we have defined, the Courant-bracket is covariant under diffeomorphisms, B-transformations and T-duality. The covariance under diffeomorphisms is manifest from the definition, while the convariance under gauge transformations of B follows from an identity proved in [4],

$$[e^{\delta B}\mathsf{A}, e^{\delta B}\mathsf{B}]_C = e^{\delta B}[\mathsf{A},\mathsf{B}]_C + i_{\pi(\mathsf{B})} i_{\pi(\mathsf{A})}\delta H , \tag{2.16}$$

where $\delta H = d\delta B$ and we define $\pi : T \oplus T^* \to T$ to be the projection onto the tangent bundle; in other words, $\pi(V + \omega) = V$. When $d\delta B = \delta H = 0$, we find

$$[e^{\delta B}\mathsf{A}, e^{\delta B}\mathsf{B}]_C = e^{\delta B}[\mathsf{A},\mathsf{B}]_C , \qquad (d\delta B = 0) \tag{2.17}$$

which is the expection result for covariance.

That the Courant-bracket is covariant under T-dualities may be surprising since there is a theorem in [4] which states that gauge transformations of B and diffeomorphisms are the only allowed symmetries. However, this theorem does not allow for the extra assumption that there are isometries.

Indeed, suppose that we have an isometry along the x direction so that the components of our generalized vectors satisfy

$$\partial_x V_i = \partial_x \omega_i = 0 \ . \tag{2.18}$$

Let μ run over the other coordinates besides x and define $W+\chi=[V_1+\omega_1,V_2+\omega_2]$. We can then expand out (2.15) to give

$$W^x = V_1^\nu \partial_\nu V_2^x - (1\leftrightarrow 2)\,, \tag{2.19}$$
$$W^\mu = V_1^\nu \partial_\nu V_2^\mu - (1\leftrightarrow 2)\,, \tag{2.20}$$
$$\chi_x = V_1^\nu \partial_\nu \omega_{2x} - (1\leftrightarrow 2)\,, \tag{2.21}$$
$$\chi_\mu = V_1^\nu \partial_\nu \omega_{2\mu} + \frac{1}{2}\partial_\mu(V^x\omega_x) + \frac{1}{2}\partial_\mu(V^\mu\omega_\mu) - (1\leftrightarrow 2)\ . \tag{2.22}$$

Note that switching $V_i^x \leftrightarrow \omega_{ix}$ switches $W^x \leftrightarrow \chi_x$ while W^μ and χ_μ are left alone. This yields our desired formula:

$$[\mathsf{T}_x(V_1+\omega_1),\mathsf{T}_x(V_2+\omega_2)]_C = \mathsf{T}_x[V_1+\omega_1,V_2+\omega_2]_C\ . \tag{2.23}$$

We emphasize again that this formula only holds when the generalized vectors are independent of the direction we are dualizing.

An interesting property of the Courant-bracket is that it does not satisfy the Jacobi identity. Rather [4],

$$[[\mathsf{V}_1,\mathsf{V}_2]_C,\mathsf{V}_3]_C + \text{cyclic} = d\,\mathrm{Nij}(\mathsf{V}_1,\mathsf{V}_2,\mathsf{V}_3)\,, \tag{2.24}$$

where the Nijenhuis operator is defined by

$$\mathrm{Nij}(\mathsf{V}_1,\mathsf{V}_2,\mathsf{V}_3) = \frac{1}{3}\langle[\mathsf{V}_1,\mathsf{V}_2]_C,\mathsf{V}_3\rangle + \text{cyclic}\ . \tag{2.25}$$

The Nijenhuis operator, as we will see later, plays an important role in defining the generalized flux.

Given a isotropic subbundle $\Omega \in \mathsf{E}$, the bundle is said to be involutive if it is closed under the Courant bracket. An important property of the Nijenhuis operator is that it vanishes on an isotropic subbundle if and only if the the subbundle is involutive [4].

3. Defining the generalized NS-NS flux

Computing the flux associated with the 3-form $H = dB$ can be thought of as having three ingredients: the flux H, the cycle Σ we wish to integrate it over and the actual integration $\int_\Sigma H$. Lifting this computation to generalized geometry requires modifying each of these notions.

3.1 The generalized p-cycle

Let's begin by extending the notion of a p-cycle. A generalized p-cycle will be given by two ingredients. The first is just an ordinary p-dimensional manifold Σ which is a submanifold of the spacetime manifold M equipped with a metric and B-field.[4]

[4]In the case of a T-fold, this definition is inadequate since M is no longer a manifold. In the cases we will consider, we will take M to be three-dimensional and the cycle Σ which wraps M to be just identified with M itself. We will not attempt to give a rigorous definition here of what it means, in general, for a T-fold to have a "sub-T-fold".

Given such a Σ, we can try to pull back the bundle $\mathsf{E} = T_M \oplus T_M^*$ to Σ. There is a slight subtlety in doing this; in the presence of a nontrivial B-field, the bundle E is twisted by a gerbe [3]. However, since we can pull back the B-field to Σ, we can just put locally $\mathsf{E}_\Sigma = T_\Sigma \oplus T_\Sigma^*$, where it is understood that globally T_Σ^* is twisted by the pullback of the B-field. We can also pull back the splitting of E into $\mathsf{C}^+ \oplus \mathsf{C}^-$. This is accomplished by pulling back g and B to Σ and then constructing the matrix G given in (2.13).

Given our 3-cycle, Σ and its associated bundle, E_Σ, we define a *generalized* 3-cycle,[5] by specifying a "frame", Ω, which we take to be a maximal isotropic subbundle $\Omega \subset \mathsf{E}_\Sigma$. Ω is to be thought of as the analogue of T^*. The idea is that if we found some set of T-dualities, diffeomorphisms and B-transformations which take Ω to T_Σ^* , then our definition of a three cycle should reduce to an ordinary three cycle and the generalized flux should reduce to the standard H-flux.

The reader might ask why we *need* to introduce Ω as an extra piece of data. The necessity of including Ω follows from the fact that *one can construct manifolds which have multiple kinds of flux.* The choice of a frame Ω is just the right extra information to select which of these fluxes we wish to measure.

To specify our choice of Ω, it will be useful to introduce a vielbein, V_i, which spans Ω and satisfies

$$\mathsf{G}(\mathsf{V}_i\,,\mathsf{V}_j) = \delta_{ij}\,, \qquad i \in \{1,2,3\}\ . \tag{3.1}$$

Such a choice of vielbein will typically not exist globally, but we will check in the flux-formula that we write down that we have an invariance under $\mathsf{V}_i \to O_i{}^j \mathsf{V}_j$ for $O \in \mathrm{SO}(p)$ so that everything depends only on Ω.

3.2 The measure for integration

In order to get some intuition for the role of the Ω in our definition of the generalized 3-cycle, consider the case when $\Omega = T_\Sigma^*$. In this case, the vielbein V_i which spans Ω is just a collection of forms;

$$\mathsf{V}_i = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}, \qquad \omega_i \in T_\Sigma^*\ . \tag{3.2}$$

The property (3.1), using the explicit form of G given in (2.13), reduces to

$$\omega_{i\mu}\omega_{j\nu}g^{\mu\nu} = \delta_{ij}\ . \tag{3.3}$$

Thus, $\omega_{i\mu}$ is an ordinary vielbein. To define a measure, we can simply wedge the V's together, giving the volume form,

$$\mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \ldots \wedge \mathsf{V}_p = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_p\ . \tag{3.4}$$

If we have coordinates $\xi^{1,2,\ldots,p}$ on our space Σ, this reduces to

$$\begin{aligned}\omega_{1\mu}\omega_{2\nu}\ldots\omega_{p\rho}\, d\xi^\mu \wedge d\xi^\nu \wedge \ldots \wedge d\xi^\rho &= \det_{i\mu}(\omega_{i\mu}) d\xi^1 \wedge d\xi^2 \wedge \ldots \wedge d\xi^p \\ &= \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge \ldots \wedge d\xi^p\,,\end{aligned} \tag{3.5}$$

[5] This definition of a generalized 3-cycle should be compared with Gualtieri's different definition of a generalized (complex) submanifold [4].

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where $g = \det(g_{\mu\nu})$. This gives us a suitable measure for integrating a scalar.

To define a measure for a general set of V_i, consider that under T-duality along, say, the ξ^1 direction, the form $d\xi^1$ would be exchanged with the vector $\partial/\partial\xi^1$. Thus an integration measure,

$$d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \,, \tag{3.6}$$

would formally become the somewhat absurd looking

$$\frac{\partial}{\partial\xi^1} \wedge d\xi^2 \wedge d\xi^3 \,. \tag{3.7}$$

This rather odd looking measure should be interpreted as telling us that one of the directions we are integrating over is actually a coordinate on the T-dual circle. Indeed, it is convenient to make the *formal* replacement,

$$\frac{\partial}{\partial\xi^\mu} \to d\hat{\xi}_\mu \,, \tag{3.8}$$

where $\hat{\xi}_\mu$ is the coordinate T-dual to ξ^μ. We can then write a vector field as $V^\mu\partial_\mu \to V^\mu d\hat{\xi}_\mu$. Our measure, (3.7), then takes the more visually appealing form,

$$d\hat{\xi}_1 \wedge d\xi^2 \wedge d\xi^3 \,. \tag{3.9}$$

Intuitively, as the notation suggests, we should integrate over $\xi^{2,3}$ using standard integration while for the ξ^1 coordinate, we should integrate over its T-dual, $\hat{\xi}_1$. Postponing until the next subsection the precise rules for doing this, we can write down the formal measure:

$$\mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \ldots \wedge \mathsf{V}_p \,, \tag{3.10}$$

which reduces to the ordinary integration measure (3.5) when the V_i take the form (3.2).

3.3 Integration over the generalized measure

To understand how we should define integration over the generalized measure, it is useful to consider the example of a generalized 1-cycle parametrized by a coordinate ξ^1 with period $\Delta\xi^1$. In this case, our vielbein is a single vector, V. Suppose that we take $\mathsf{V} = V$ where V is a one component vector. We would like to define

$$\int \mathsf{V} = \int V \,. \tag{3.11}$$

Using (3.1) implies that

$$(V^1)^2 g_{11} = 1 \,, \tag{3.12}$$

so that we may take

$$V = \frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial\xi^1} \,. \tag{3.13}$$

Using the notation introduced in (3.8) and assuming that g_{11} does not depend on ξ^1, we can write this as

$$V = \sqrt{\hat{g}_{11}}\, d\hat{\xi}_1 \,, \tag{3.14}$$

where $\hat{g}_{11} = g_{11}^{-1}$ is the metric of the dual circle as found from the Buscher rules. It is now clear how we can integrate over V. We put

$$\int V = \int \sqrt{\hat{g}_{11}}\, d\hat{\xi}_1 = \hat{L}\,, \tag{3.15}$$

where $\hat{L}$ is the length of the dual circle. Noting that $\hat{L} = L^{-1}$ where $L = \sqrt{g_{11}}\Delta\xi^1$ is the length of the ξ^1 circle, we learn that

$$\int d\hat{\xi}_1 \equiv \frac{1}{\Delta\xi^1}\,, \tag{3.16}$$

which is just what one would expect for the period of the dual circle. This is the basic definition that will allow us to integrate over the generalized measure.

Note that it was important that our integrand did not depend on the direction we were integrating over. It would be very interesting if there were a natural definition of

$$\int f(\xi) d\hat{\xi} = ?\,, \tag{3.17}$$

but we suspect that, in general, no such definition exists. Instead we will insist that whenever we have an integral of the form,

$$\int f(\xi^\mu)\, d\xi^1 \wedge \ldots \wedge d\xi^q \wedge d\hat{\xi}_{q+1} \wedge \ldots \wedge d\hat{\xi}_p\,, \tag{3.18}$$

that $f(\xi^\mu)$ only depends on $\xi^{1,2,\ldots,q}$ and that $\xi^{q+1,\ldots,p}$ are periodically identified with period $\Delta\xi^i$. We can then repeatedly apply formula (3.16) to yield

$$\left(\prod_{i=q+1}^{p} \frac{1}{\Delta\xi^i}\right) \int f(\xi^\mu)\, d\xi^1 \wedge \ldots \wedge d\xi^q\ . \tag{3.19}$$

This reduces the rather mysterious looking integral (3.18) to an ordinary integral. Effectively, we are *dimensionally reducing* along the circle directions $\xi^{q+1,\ldots,p}$, which we can think of as being fibered over the $\xi^{1,\ldots,q}$ directions. After dimensional reduction, we can then consistently integrate over the base space directions, $\xi^{1,\ldots,q}$.

3.4 The fiber condition

To complete our definition of integration, we wish to impose that the integral over the generalized measure can always be reduced to an ordinary integral by repeated application of the rule (3.16). In this subsection we give a simple criteria for this requirement which we will refer to as the *fiber condition.* It is quite likely that a more general integration can be defined, but the condition given will suffice for our purposes.

To ensure that whenever we have an integral over a dual direction, the associated coordinate parametrizes a circle, we insist that we can write Σ as a T^{p-q} with coordinates $\xi^{q+1,\ldots,p}$ fibered over a space with coordinates $\xi^{1,\ldots,q}$. We then insist that the non-zero vectors $\pi(\mathsf{V}_i)$ are a basis for the tangent bundle of the torus fiber, while the forms live on

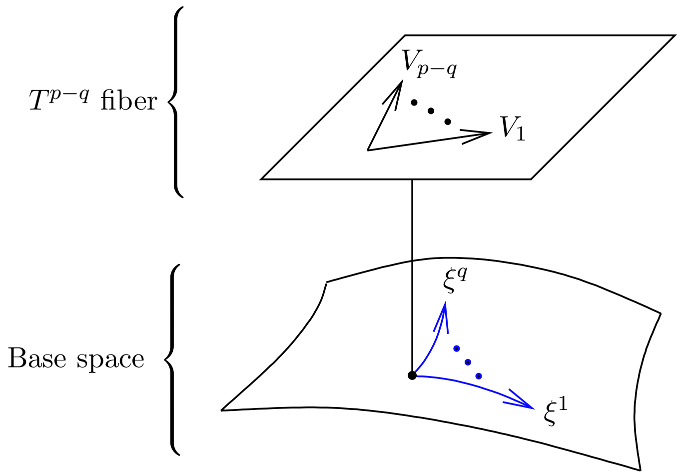

Figure 1: In order to define integration over our generalized p-cycle, we demand that it take the form of a T^{p-q} fibered over a base space. We further demand that nothing depend on the coordinates of the torus and that the vector parts of the generalized vielbein span the tangent bundle of the T^{p-q}. Finally, we assume that the forms live in the span of the $d\xi^i$ for $i \in \{1, \ldots, q\}$.

the base space. Furthermore, we impose that nothing depends on the T^{p-q} fiber. These rules, which together form the fiber condition are illustrated figure 1.

Having imposed such a strong condition on our generalized cycles, we can ask: to what extent is the fiber condition invariant under diffeomorphisms, B-transformations and T-duality? Already with the diffeomorphisms we see that we should restrict the diffeomorphisms to those which preserve the torus fiber and do not depend on the torus directions. Generically, other diffeomorphisms will exist, but these will take us away from the space of generalized cycles where we know how to integrate.

When we study B-transformations, it is clear that we should not allow those transformations that depend on the torus coordinates. In addition, recalling that the B-transformations take the form,

$$e^{\delta B}(V + \omega) = V + \omega + i_V \delta B\,, \tag{3.20}$$

we see that for all $V \in T_{T^{p-q}}$ we should demand that $i_V \delta B$ is a form on the base space.[6] This ensures, for instance, that a measure of the form, $\prod V_i \wedge \prod \omega_j = \prod V_i^\mu d\hat{\xi}_\mu \wedge \prod \omega_{j\mu} d\xi^\mu$, will be invariant under the restricted B-transformations, since the shift in V will be some linear combination of the ω_i, which will vanish when wedged with $\prod_j \omega_j$.

Finally, we can ask when the fiber condition is invariant under T-duality. First, suppose we T-dualize along one of the directions of the fiber. In this case, T-duality simply removes one of the directions of the fiber and adds it to the base, taking $T^{p-q} \to T^{p-q-1}$. That the

[6]Because of this restriction on the B-transformations, the fiber condition is not invariant under the expected $\mathrm{SO}(p-q, p-q; \mathbb{Z})$ symmetry of the T^{p-q} torus fiber. In order to restore this invariance one must define a notion of integration that allows the dual coordinates to mix with the coordinates in the fiber directions.

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integral remains invariant follows by construction from our definition of integration along the vector-like directions.

We can also T-dualize along a base-space direction. Suppose that the direction we wish to T-dualize along is generated by a vector V. Then, provided that V doesn't depend on the fiber coordinates, it follows that we can, at least locally, define a new coordinate associated with the isometry, while leaving the fiber coordinates alone. Under T-duality, this coordinate is added to the fiber coordinates to take $T^{p-1} \to T^{p-q+1}$. That the integral is unchanged follows again from our definitions.

Note that the fiber condition implies the weaker conditions,

$$\langle \mathsf{V}_i, \mathsf{V}_j \rangle = 0\,, \tag{3.21}$$

$$[\mathsf{V}_i, \mathsf{V}_j]_C = 0\ . \tag{3.22}$$

These conditions are very natural since they are trivially satisfied in the ordinary integration case when $\Omega = T^*_\Sigma$. They are not, however, sufficient to ensure that one can perform the generalized integral.

3.5 The generalized version of NS-NS flux

Having defined a generalized integral and a generalized 3-cycle, we must now write down the flux that we wish to integrate over. The result, as given in the introduction, which will be motivated by the examples, is given by

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = -\,\mathrm{Nij}(\widetilde{\mathsf{V}}_1, \widetilde{\mathsf{V}}_2, \widetilde{\mathsf{V}}_3)\,, \tag{3.23}$$

where the Nijenhuis operator was defined in (2.25) and we define

$$\widetilde{\mathsf{V}} = \mathsf{G}\mathsf{V}\ . \tag{3.24}$$

The complete formula for the flux is

$$\int_\Sigma \mathsf{H} \equiv \int_\Sigma \mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3)\ \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3\ . \tag{3.25}$$

Since Ω is defined to be a isotropic bundle, it follows that $\widetilde{\Omega} = \mathsf{G}\Omega$ is also isotropic. Using the result of Gualtieri [4] that Nij on an isotropic subbundle is actually a tensor, we learn that

$$\mathsf{H}(\mathsf{V}_i, \mathsf{V}_j, O_k{}^m \mathsf{V}_m) = O_m{}^k \mathsf{H}(\mathsf{V}_i, \mathsf{V}_j, \mathsf{V}_k) \tag{3.26}$$

for any matrix $O_i{}^j$. Writing

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3)\ \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 = \frac{1}{3!}\sum_{i,j,k} \mathsf{H}(\mathsf{V}_i, \mathsf{V}_j, \mathsf{V}_k)\ \mathsf{V}_i \wedge \mathsf{V}_j \wedge \mathsf{V}_k, \tag{3.27}$$

we see that the flux is invariant under rotations $\mathsf{V}_i \to O_i{}^J \mathsf{V}_j$ provided that $O \in \mathrm{SO}(3)$. Hence, our flux formula only depends on Ω and not on any particular basis.[7]

[7] Indeed, (3.27) can be defined as the projection of the Nij operator onto $\widetilde{\Omega}$ using metric $\mathsf{G}(,)$. This defines an element of $\wedge^3\widetilde{\Omega} \in \wedge^3\mathsf{E}$.

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4. Special cases of the generalized-flux formula

In this section we apply the flux formula (3.25) to various specific cases in order to show that it reproduces standard examples. To do so, it is useful to have an explicit expression for H in terms of the components of the vielbein, V_i . Let our vielbein take the form,

$$\mathsf{V}_i = \begin{pmatrix} V_i \\ \omega_i \end{pmatrix} . \tag{4.1}$$

We denote

$$\widetilde{\mathsf{V}}_i = \mathsf{G}\mathsf{V}_i = \begin{pmatrix} \widetilde{V}_i \\ \widetilde{\omega}_i \end{pmatrix} = \begin{pmatrix} g^{-1}\omega_i - g^{-1}BV_i \\ gV_i - Bg^{-1}BV_i + Bg^{-1}\omega_i \end{pmatrix} . \tag{4.2}$$

Rather than substituting (4.2) directly into the formula for H and expanding it out in terms of V_i and ω_i it is more useful to take the following approach: Note that

$$\widetilde{\mathsf{V}}_i = e^B \begin{pmatrix} \widetilde{V}_i \\ gV_i \end{pmatrix} . \tag{4.3}$$

We can then use (2.16) to find

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = \widetilde{V}_1^\mu \widetilde{V}_2^\nu \widetilde{V}_3^\rho H_{\mu\nu\rho} + \left[\widetilde{V}_1^\mu \widetilde{V}_2^\nu (V_{3\nu,\mu} - V_{3\mu,\nu}) + \text{cyclic}\right] - \left[\widetilde{V}_1^\mu \partial_\mu (\widetilde{V}_{[2}^\nu V_{3]\nu}) + \text{cyclic}\right] , \tag{4.4}$$

where in each of the terms in square brackets we must add the cyclic permutations of 123 and the indices are raised and lowered with g. With this formula in hand, we turn to the special cases.

4.1 Three-form flux

The simplest case we can examine is when $\Omega = T^*$ and our vielbein takes the form,

$$\mathsf{V}_i = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} . \tag{4.5}$$

In this case the property (3.1) reduces to $\omega_{i\mu}\omega_{j\nu}g^{\mu\nu} = \delta_{ij}$, which implies that the ω_i form a vielbein in the ordinary sense. We also have

$$\widetilde{\mathsf{V}}_i = \begin{pmatrix} g^{-1}\omega_i \\ Bg^{-1}\omega_i \end{pmatrix} , \tag{4.6}$$

so that $\widetilde{V}_i = \omega_i^\mu$. Since $V_i = 0$, formula (4.4) gives

$$\mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3) = \omega_1^\mu \omega_2^\nu \omega_3^\rho H_{\mu\nu\rho} . \tag{4.7}$$

Hence, the flux integral becomes

$$\int_\Sigma \mathsf{H}(\mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3)\, \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 = \int_\Sigma (\omega_1^\mu \omega_2^\nu \omega_3^\rho H_{\mu\nu\rho})\, \omega_{1\gamma}\omega_{2\beta}\omega_{3\tau}\, d\xi^\gamma \wedge d\xi^\beta \wedge d\xi^\tau . \tag{4.8}$$

Noting again that the ω_i form an ordinary vielbein, this reduces to

$$\int_\Sigma H\,, \tag{4.9}$$

which is the standard formula for 3-form flux. Note that we did not need to worry about the fiber condition since $V_i = 0$.

4.2 Geometric flux

Geometric fluxes arise from T-dualizing spaces with H-flux. We suppose that Σ has one killing vector V that generates a circle bundle. We then pick as our basis for Ω,

$$\mathsf{V}_{1,2} = \begin{pmatrix} 0 \\ \omega_{1,2} \end{pmatrix}, \qquad \mathsf{V}_3 = \begin{pmatrix} V \\ BV \end{pmatrix}, \tag{4.10}$$

where the ω are a complete set of forms on the base of the circle bundle. One can also take V_3 to be a pure vector; however the above choice makes it clear that Ω is a global section of E when there is a non-trivial B-field. Note that we have

$$\widetilde{\mathsf{V}}_3 = \begin{pmatrix} 0 \\ gV \end{pmatrix}, \tag{4.11}$$

so that $\widetilde{V}_3^\mu = 0$. The vielbein property (3.1) becomes

$$V^\mu V^\nu g_{\mu\nu} = 1\,, \qquad \omega_{i\mu}\omega_{j\nu}g^{\mu\nu} = \delta_{ij}\ . \tag{4.12}$$

It is convenient to pick one of our coordinates, ξ^3 to be the circle coordinate with period 1, so that, using (4.12), we have

$$V = \frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial\xi^3}\ . \tag{4.13}$$

We can now use the formula (4.4) to compute the flux,

$$\mathsf{H} = \omega_1^\mu\omega_2^\nu(V_{\nu,\mu} - V_{\mu,\nu}) + \frac{1}{2}(\omega_1^\mu(\omega_2(V_3))_{,\mu} + (1 \leftrightarrow 2))\ . \tag{4.14}$$

However, using that Ω must be isotropic, we have that $\omega_{1,2}(V) = 0$, and second term vanishes. Thus, we get just

$$\mathsf{H} = \omega_1^\mu\omega_2^\nu(V_{\nu,\mu} - V_{\mu,\nu})\ . \tag{4.15}$$

Our measure factor $\mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3$ gives

$$\omega_{1\mu}\omega_{2\nu}B_{\rho\tau}V_3^\tau d\xi^\mu \wedge d\xi^\nu \wedge d\xi^\rho + \frac{1}{\sqrt{g_{33}}}\omega_{1\mu}\omega_{2\nu}d\xi^\mu \wedge d\xi^\nu \wedge d\hat{\xi}_3\ . \tag{4.16}$$

The first term, however vanishes again by the isotropy of Ω.[8] Hence, we find just

$$\frac{1}{\sqrt{g_{33}}}\omega_{1\mu}\omega_{2\nu}d\xi^\mu \wedge d\xi^\nu \wedge d\hat{\xi}_3\ . \tag{4.17}$$

[8] *Proof*: We are imposing that $\omega_{1,2}(V_3) = 0$. We also have $\omega'_\mu = B_{\mu\nu}V^\nu$ satisfies $\omega'(V) = 0$. However, the space of forms α satisfying $\alpha(V) = 0$ is only two dimensional. Thus ω' is a linear combination of $\omega_{1,2}$ and $\omega_1 \wedge \omega_2 \wedge \omega' = 0$.

Putting everything together, our flux takes the form,

$$\int \frac{1}{\sqrt{g_{33}}} \omega_1^\mu \omega_2^\nu (V_{\nu,\mu} - V_{\mu,\nu})\, \omega_{1\mu}\omega_{2\nu} d\xi^\mu \wedge d\xi^\nu \wedge d\hat{\xi}_3 \; . \tag{4.18}$$

Again using the condition that $\omega_{1,2}(V) = 0$, this can be rewritten as

$$\int \omega_1^\mu \omega_2^\nu (dA)_{\mu\nu}\, \omega_{1\mu}\omega_{2\nu} d\xi^\mu \wedge d\xi^\nu \wedge d\hat{\xi}_3 \, , \tag{4.19}$$

where

$$A_\mu = (\sqrt{g_{33}})^{-1} V_\mu = \frac{g_{\mu 3}}{g_{33}} \; . \tag{4.20}$$

Notice that A_μ is just the connection on the circle bundle generated by V. Examining the Buscher-rules, one notes that it is also the T-dual of $B_{\mu 3}$.

Since we have picked the length of our circle-coordinate to be one, we may simply drop the $d\hat{\xi}_3$. The integral then reduces to

$$\int \omega_1^\mu \omega_2^\nu (dA)_{\mu\nu}\, \omega_{1\mu}\omega_{2\nu}\, d\xi^\mu \wedge d\xi^\nu = \int dA \, , \tag{4.21}$$

where the integral is performed over the base of the circle-fibration. This gives the first Chern-class of the circle bundle, which is the geometric flux.

Example: The f-space. As a simple example, consider the pure-metric space given by,

$$ds^2 = (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3 + n\xi^1 d\xi^2)^2 \, , \tag{4.22}$$

The $\xi^{2,3}$ directions can be compactified in the usual way under the symmetries $\xi^{2,3} \to \xi^{2,3} + 1$. The ξ^1 direction can also be compactified, but under the combined symmetry,

$$\xi^1 \to \xi^1 + 1 \, , \qquad \xi^2 \to \xi^2 \, , \qquad \xi^3 \to \xi^3 - n\xi^2 \; . \tag{4.23}$$

This space is known variously as a Scherk-Schwarz compactification, twisted torus and nil-manifold as well just the f-space [33, 14, 9, 34].

The vielbein appropriate for measuring the geometric flux takes a very simple form in this space;

$$\mathsf{V}_1 = d\xi^1 \, , \qquad \mathsf{V}_2 = d\xi^2 \, , \qquad \mathsf{V}_3 = \frac{\partial}{\partial \xi^3} \; . \tag{4.24}$$

Here we are treating the circle bundle as the ξ^3 direction. Inspecting the metric (4.22), we see that $A = n\xi^1 d\xi^2$ and $dA = n d\xi^1 \wedge d\xi^2$. Formula (4.21) reduces to just

$$\int n \, d\xi^1 \wedge d\xi^2 = n \, , \tag{4.25}$$

which gives the geometric-flux. It is important to note that this integral should not be thought of as being performed over a 2-dimensional slice of the f-space. In fact, no such slice exists. The reader may check, for example that the plane determined by $\xi^3 = 0$ is not consistent with the identification (4.23). Rather, after we perform the "integral" over

$\partial/\partial\xi^3$ we have effectively dimensionally reduced along the ξ^3-direction so that each point specified by $\xi^{1,2}$ corresponds to circle.

Note that T-dualizing along the ξ^3 direction gives a space with metric and B-field,

$$ds^2 = (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2\,, \qquad B = n\xi^1\, d\xi^2 \wedge d\xi^3\ . \tag{4.26}$$

Applying the same T-duality to the vielbein, (4.24) becomes

$$\mathsf{V}_1 = d\xi^1\,, \qquad \mathsf{V}_2 = d\xi^2\,, \qquad \mathsf{V}_3 = d\xi^3\ . \tag{4.27}$$

Thus, to compute the generalized flux we should use (4.9) which gives $\int dB\, d^3\xi = n$, demonstrating the expected invariance under T-duality.

4.3 Non-geometric Q-flux

One kind of non-geometric flux, known as Q-flux, which has been studied recently [14, 9, 11, 10, 31, 18, 26, 29, 30, 15, 17] is associated with a so-called β-transformation. A β-transformation is the double T-dual of a B-transformation. It acts on generalized vectors as

$$\mathsf{V} \to e^{\beta}\mathsf{V} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V \\ \omega \end{pmatrix}, \tag{4.28}$$

where β is an antisymmetric matrix.

A Q-space is a T^2 fibered over an S^1 in which, when one goes around the S^1, one performs a β-transformation. In order to find global sections of E_Σ, we should look for a vielbein that is not affected by β-transformations. Examining the form of the β-transformation given in (4.28), we see that generalized vectors whose 1-form part vanishes are unaffected by β-transformations.

For definiteness, let our space be a T^2 with coordinates $\xi^{2,3}$ fibered over an S^1 with coordinate ξ^1. Consider a metric and B-field of the form,

$$g = \begin{pmatrix} 1 & \\ & g_{ab} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \\ & B_{ab} \end{pmatrix}, \tag{4.29}$$

where a, b run over 2, 3 and nothing depends on the coordinates $\xi^{2,3}$ of the T^2. We take Ω to be spanned by

$$\mathsf{V}_1 = d\xi^1\,, \qquad \mathsf{V}_2 = v_1^a \frac{\partial}{\partial \xi^a}\,, \qquad \mathsf{V}_3 = v_3^a \frac{\partial}{\partial \xi^a}\ . \tag{4.30}$$

The property (3.1) is now quite complicated:

$$v_i^a (g - Bg^{-1}B)_{ab} v_j^b = \delta_{ij}\ . \tag{4.31}$$

However, it is straightforward to find an appropriate pair of v's and substitute it into the general formula (3.25). This yields

$$\mathsf{H}\ \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 = \left[\frac{\partial}{\partial \xi^1} \mathrm{Re}\left(\frac{1}{\tau}\right)\right] d\xi^1 \wedge d\hat{\xi}_2 \wedge d\hat{\xi}_3\,, \tag{4.32}$$

where we have defined $\tau = B_{12} + i\sqrt{g}$. Assuming that the $\xi^{2,3}$ coordinates run from 0 to 1, we can perform the integral over them trivially, yielding

$$Q\text{-flux} = \int \mathsf{H}\ \mathsf{V}_1 \wedge \mathsf{V}_2 \wedge \mathsf{V}_3 = \int d\xi^1 \frac{\partial}{\partial \xi^1} \mathrm{Re}\left(\frac{1}{\tau}\right) . \tag{4.33}$$

To illuminate the meaning of this expression, we note that a β-transformation acts as

$$\tau \to \frac{\tau}{1 + \beta\tau} , \tag{4.34}$$

which takes

$$\mathrm{Re}\left(\tau^{-1}\right) \to \mathrm{Re}\left(\tau^{-1}\right) + \beta . \tag{4.35}$$

Since the formula (4.33) is an integral of a total derivative, the Q-flux is given by the β-transformation that maps the top to the bottom. Since (4.34) must be an element of $\mathrm{SL}(2, \mathbb{Z})$, this gives an integer.

Example: The standard *Q*-space. The original example of a space with Q-flux is found by T-dualizing the f-space example (4.22) along the ξ^2-direction [9]. This gives a metric and B-field,

$$ds^2 = (d\xi^1)^2 + \frac{1}{1 + n^2(\xi^1)^2}\left((d\xi^2)^2 + (d\xi^3)^2\right) , \qquad B = \frac{n\xi^1}{1 + n^2(\xi^1)^2} d\xi^2 \wedge d\xi^3 . \tag{4.36}$$

The $\xi^{2,3}$ directions are compactified with unit period, while the ξ^1 direction is compactified with unit period only up to a β-transformation. The appropriate vielbein is given by T-dualizing the vielbein in (4.24) yielding

$$\mathsf{V}_1 = d\xi^1 , \qquad \mathsf{V}_2 = \frac{\partial}{\partial \xi^2} , \qquad \mathsf{V}_3 = \frac{\partial}{\partial \xi^3} . \tag{4.37}$$

Substituting these into the flux formula yields

$$\int_\Sigma \mathsf{H} = \int n\, d\xi^1 \wedge d\hat{\xi}_2 \wedge d\hat{\xi}_3 = n . \tag{4.38}$$

To see that this agrees with the more general formula (4.33) note that

$$\tau = \frac{n\xi^1 + i}{1 + n^2(\xi^1)^2} = \frac{1}{n\xi^1 - i} . \tag{4.39}$$

Hence,

$$\frac{\partial}{\partial \xi^1} \mathrm{Re}(\tau^{-1}) = n , \tag{4.40}$$

which reproduces (4.38).

5. The generalized connection and the flux

In this section, we discuss a generalized connection that acts on generalized vectors and its relation to the generalized flux. Although the flux H often arises as a torsion of a connection, computing the analogue of the torsion of the generalized connection, we see that it vanishes. However, we find that the flux H arises from an object very similar to the torsion.

For clarity, it is useful to introduce an index notation. We denote a generalized vector by V^I where the index I runs over the tangent indices followed by the cotangent indices. The indices can be raised and lowered using the metric,

$$\mathsf{X}_{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{X}^{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.1}$$

Note that a lowered index, as in V_I simply runs over the cotangent indices first followed by the tangent indices. The matrix G has index structure $\mathsf{G}^I{}_J$. The lowered matrix G_{IJ} is the positive definite metric introduces in (2.14). The raised matrix G^{IJ} is, as one would like, the inverse of G_{IJ} so that $\mathsf{G}_{IJ}\mathsf{G}^{JK} = \delta_I^K$. This follows from the basic property that $\mathsf{G}^2 = 1$.

The goal of this section is to write down a covariant derivative D^I which, when acting on vectors,

$$\mathsf{D}^I \mathsf{V}^J \,, \tag{5.2}$$

gives a two index object covariant under diffeomorphisms, B-transformations and T-duality. Note that this is not a connection in the ordinary sense, since it allows one to take derivatives with respect to the T-dual coordinates.

To define the generalized connection, we begin by defining an ordinary connection on E. This connection will be invariant under diffeomorphisms and B-transformations, but will not be invariant under T-duality.

We take the connection to be of the form

$$D_\mu = \partial_\mu + \Omega_\mu \,, \tag{5.3}$$

where Ω_μ is a matrix $\Omega_\mu{}^I{}_J$ which acts on the generalized vector indices. When the B-field vanishes, it is very natural to take the connection to have the form

$$D_\mu \Big|_{B=0} = \begin{pmatrix} \nabla_\mu & 0 \\ 0 & \hat{\nabla}_\mu \end{pmatrix}, \tag{5.4}$$

where ∇_μ is the Levi-Civita connection on vectors and $\hat{\nabla}_\mu$ is the Levi-Civita on 1-forms.

When $B \neq 0$, one can partially fix the form of D_μ by demanding that it annihilate both X^{IJ} and G^{IJ} and that it transform covariantly under B-transformations. This unfortunately is not enough to completely determine the connection, as one is still left with a one-parameter family of possible connections:

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \left[\begin{pmatrix} \nabla_\mu & 0 \\ 0 & \hat{\nabla}_\mu \end{pmatrix} + a \begin{pmatrix} 0 & \frac{1}{2} g^{-1} H_\mu g^{-1} \\ \frac{1}{2} H_\mu & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}. \tag{5.5}$$

Here we have used the shorthand H_μ for $H_{\mu\nu\rho}$ where ν and ρ are treated as matrix indices. To fix an appropriate choice for a, it is useful to turn to string theory for guidance. Recall that in the fermionic terms of the $N = 1$ string action the kinetic terms use the connection

$$\nabla_\mu^\pm = \nabla_\mu \pm \frac{1}{2} g^{-1} H_\mu \,, \tag{5.6}$$

where we take + for the right moving fermions and − for the left moving fermions. This connection, known as the *Bismut connection* in the generalized literature, was first introduced in string theory by Gates, Hull and Rocek [35] and is relevant for a number of applications in generalized complex geometry [4, 3].

We can now fix the form of (5.5) by insisting that if $\mathsf{V} \in \mathsf{C}^\pm$ that

$$\pi(D_\mu \mathsf{V}) = \nabla_\mu^\pm \pi(\mathsf{V}) \ . \tag{5.7}$$

In other words, the covariant derivative just acts as the Bismut connection on the vector part of V. This extra condition fixes $a = 1$ and gives the lift of the Bismut connection to generalized geometry:

$$D_\mu \equiv \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} \nabla_\mu & \frac{1}{2} g^{-1} H_\mu g^{-1} \\ \frac{1}{2} H_\mu & \hat{\nabla}_\mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} . \tag{5.8}$$

This connection has nice properties under T-duality. Suppose the x-direction parametrizes a circle and that neither g nor B depends on x. Then we find[9]

$$D^\mu \xrightarrow{\mathrm{T}_x} \mathsf{T}_x D^\mu \mathsf{T}_x \,, \qquad \mu \neq x \,, \tag{5.9}$$

$$D^x \xrightarrow{\mathrm{T}_x} \mathsf{T}_x \left(B_{x\sigma} D^\sigma - \mathsf{G} g_{x\sigma} D^\sigma \right) \mathsf{T}_x \ . \tag{5.10}$$

The matrices T_x are the elements of $\mathrm{SO}(d, d)$ which represent T-duality along the x direction. Since T-duality switches vectors with forms, (5.10) gives us the form part of the generalized connection. Indeed, setting

$$\mathsf{D}^I = \begin{pmatrix} D^\mu \\ -\mathsf{G} D_\mu + B_{\mu\sigma} D^\sigma \end{pmatrix} , \tag{5.11}$$

it is straightforward to check using (5.9) and (5.10) that, when acting on a vector, as in $\mathsf{D}^I \mathsf{V}^J$, that the resulting two index object transforms covariantly under diffeomorphisms, B transformations and T-duality.

5.1 Parallel transport and torsion

We define the parallel transport of V_2 along V_1 by

$$\mathsf{D}_{\mathsf{V}_1} \mathsf{V}_2^K = \mathsf{X}_{IJ} \mathsf{V}_1^I \, \mathsf{D}^J \mathsf{V}_2^K \ . \tag{5.12}$$

[9]We have not found a simple proof of this formula. However, it is straightforward, although cumbersome, to check by a direct application of the Buscher-rules.

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This definition of parallel transport has a nice formula in terms of the connection (5.8), which can be found using the definitions (5.11) and (5.12):

$$\mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2^K = \pi(\widetilde{\mathsf{V}}_1)^\mu D_\mu \mathsf{V}_2^K - \pi(\mathsf{V}_1)^\mu D_\mu \widetilde{\mathsf{V}}_2^K \; . \tag{5.13}$$

This expression gives a derivation of the "Leibniz rule",

$$\mathsf{D}_{\mathsf{V}_1}(f\mathsf{V}_2) = f\mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2 + [\pi(\widetilde{\mathsf{V}}_1)(f)]\mathsf{V}_2 - [\pi(\mathsf{V}_1)(f)]\widetilde{\mathsf{V}}_2 \; . \tag{5.14}$$

Now that we have defined parallel transport, we may attempt to define a torsion

$$\mathsf{T}(\mathsf{V}_1,\mathsf{V}_2) = \mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2 - \mathsf{D}_{\mathsf{V}_2}\mathsf{V}_1 - [\mathsf{V}_1,\mathsf{V}_2] \; . \tag{5.15}$$

A nice choice for the bracket, $[,]$, which makes T into a tensor, is given by

$$[\mathsf{V}_1,\mathsf{V}_2] = \mathsf{G}[\widetilde{\mathsf{V}}_1,\widetilde{\mathsf{V}}_2]_C - \mathsf{G}[\mathsf{V}_1,\mathsf{V}_2]_C \; . \tag{5.16}$$

A straightforward, but tedious computation of $\mathsf{T}(\mathsf{V}_1,\mathsf{V}_2)$ reveals that

$$\mathsf{T}(\mathsf{V}_1,\mathsf{V}_2) = 0 \, , \tag{5.17}$$

so that, in the sense defined by (5.15) and (5.16), the torsion vanishes.

This computation suggests that the notion that the flux is given by the torsion of the connection, as holds for the Bismut connection for example, is not quite right. Consider, however, the torsion-like quantity,[10]

$$-\frac{1}{3}\left(\langle(\mathsf{D}_{\mathsf{V}_1}\mathsf{V}_2),\mathsf{V}_3\rangle - \langle(\mathsf{D}_{\mathsf{V}_2}\mathsf{V}_1),\mathsf{V}_3\rangle\right) + \text{cyclic} \; . \tag{5.18}$$

Using (5.15) and (5.16) this reduces to

$$\mathsf{H}(\mathsf{V}_1,\mathsf{V}_2,\mathsf{V}_3) - \frac{1}{3}[\langle[\mathsf{V}_1,\mathsf{V}_2]_C,\widetilde{V}_3\rangle + \text{cyclic}] \; . \tag{5.19}$$

Notice that for V's which are appropriate for a generalized 3-cycle, we would have $[\mathsf{V}_i,\mathsf{V}_j]_C = 0$, so that (5.19) would reduce to just $\mathsf{H}(\mathsf{V}_1,\mathsf{V}_2,\mathsf{V}_3)$. This is, in fact, how we originally found the flux formula.

5.2 Differentiation of tensors

Although they are not relevant for the main line of discussion, we end this section with a few observations about the action of generalized connection on tensors. For A a generalized vector, we have the following identity,

$$\mathsf{G}^I{}_J\mathsf{D}^J\mathsf{A}^K + \mathsf{D}^I(\mathsf{G}^K{}_L\mathsf{A}^L) = 0 \; . \tag{5.20}$$

This implies that the index on the generalized connection lives in the opposite half of the splitting as the index of the vector it is differentiating.

[10] For the Bismut connection, for example, we would find $\nabla^\pm_{V_1}V_2^\mu V_{3\mu} - \nabla^\pm_{V_2}V_1^\mu V_{3\mu} = \pm H(V_1,V_2,V_3) + [V_1,V_2]^\mu V_{3\mu}$, yielding the torsion plus a term that vanishes provided $[V_1,V_2]=0$.

Because of the G in the definition (5.11), D^I does not satisfy the Leibniz rule when acting on products of vectors unless all of the vectors live in C^+ or all live in C^- (in which case one can replace G by ± 1). This implies that it is not meaningful to speak of differentiating a tensor $\mathsf{T}^{I_1 I_2 \ldots I_n}$ unless it satisfies

$$\forall i,j \qquad \mathsf{G}^{I_i}{}_J \mathsf{T}^{I_1 \ldots I_{i-1} J I_{i+1} \ldots I_n} = \mathsf{G}^{I_j}{}_J \mathsf{T}^{I_1 \ldots I_{j-1} J I_{j+1} \ldots I_n} \,. \tag{5.21}$$

Because of the rule (5.20) we cannot act multiple times with the connection since the property (5.21) is not preserved under differentiation. This makes it very difficult to construct a curvature of the generalized connection.

6. Discussion

We conclude with a few comments on future directions and problems that we believe deserve further study.

1. In our construction of the generalized flux integral, we relied heavily on what we called fiber condition. This condition was required to ensure that we could give a sensible definition of integration over a generalized 3-cycle. It seems likely, however, that the spaces on which integration is well-defined could be extended. Currently, for example, our definition is not broad enough to handle spaces where coordinates and dual coordinates mix on the torus fibers and we are, thus, not able to realize a full $\mathrm{SO}(d,d;\mathbb{Z})$ symmetry for our definition of integration. This makes it difficult to understand fluxes on spaces for which there is no geometric dual.

2. Although in the examples we were able to show that the integral of H over a generalized 3-cycle was always a topological quantity and in fact an integer, it would be nice to have proof of this in the framework of generalized geometry.

3. Our discussion of the generalized connection seems far from complete. There is already a well-established connection which acts on pure spinors [4], and it would be interesting to try to connect the two. It also seems quite interesting to try to understand whether there is a natural notion of the curvature of the connection.

4. It would be nice to give a stringy derivation of the flux formula. The string action can already be written in a generalized form, following [32, 16, 17, 15], and the related works [36 – 40],

$$S = \frac{1}{4} \int \mathsf{G}_{IJ} \mathsf{Z}^I \wedge * \mathsf{Z}^J + \mathsf{B}_{IJ} \mathsf{Z}^I \wedge \mathsf{Z}^J \,, \tag{6.1}$$

   where B_{IJ} is the canonical anti-bracket of E given by the matrix,

$$\mathsf{B}_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

   and $\mathsf{Z} = dX^\mu + \Omega_\mu$ where Ω_μ is an auxiliary one-form on the worldsheet as well as in spacetime. It would be quite nice if we could replace the B term with a WZW-term involving H, but we have not yet found a way to do so.

Acknowledgments

We would like to thank Thomas Grimm, Akikazu Hashimoto, Albrecht Klemm, Leopoldo Pando Zayas, Jessie Shelton, Bernardo Uribe and Diana Vaman for useful conversations. This work was supported in part by DOE grant DE-FG02-95-ER40896 and funds from the University of Wisconsin-Madison.

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